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Gauge equivalence of σ models with non-compact Grassmannian manifolds

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Abstract. The gauge equivalence (GE) of σ models associated with non-compact Grassmannian manifolds is investigated with emphasis on the necessary restrictions for the choice of gauge elements in such cases. The importance of GE in solving a non-linear system with the help of inverse scattering data of its gauge related counterpart is demonstrated. The gauge relations between generalised Landau-Lifshitz (LL) and non-linear Schrödinger (NLS) type equations and also between non-linear σ models and generalised 'sine-sinh-Gordon' equations for non-compact $SU(p, q)/S(U(u, v)\times U(s, t))$ manifolds are established. Using H-gauge invariance of LL the GE is extended to some higher-order specific non-linear systems. The gauge connection among various LL and NLS equations are schematically represented. Along with the recovery of earlier results important new results, some with significant non-compact structures, are discovered.

1. Introduction

The gauge connection between different non-linear dynamical systems and the corresponding σ models has received much attention in the last few years. Among such systems the gauge equivalent Landau-Lifshitz equation (LLE) and non-linear Schrödinger equations (NLSE) and also non-linear σ model and sine-Gordon (SG) equations deserve special attention due to their physical relevance. After the pioneering work of Lakshmanan (1977) and Zakharov and Takhtajan (1979) the gauge relation between the first pair of systems has been extended to higher-order unitary groups (Orfanidis 1980), to compact manifolds (Honnerkamp 1981) and also to unconstrained $SU(N)$ models (Sasaki and Ruijgrok 1982). The relation between SG and σ models is due to Pohlmayer (1976) and Lund and Regge (1976) and was also reformulated in the standard form and extended to CP^N by Eichenherr and Honnerkamp (1981). Besides the non-linear systems mentioned above, there are various other types of integrable equations, e.g. NLSE of the repulsive type (Zakharov and Shabat 1973), 'attractive-repulsive' NLSE (Makhankov 1981), the sinh-Gordon equation (Podgribkov 1982) and also systems like the derivative NLSE (Kaup and Newell 1978) and mixed NLSE (Wadati *et al* 1979, Chen *et al* 1979, Gerdjikov and Ivanov 1982) equations. The aim of this investigation is to generalise the gauge equivalence scheme to the non-compact Grassmannian manifold $SU(p, q)/S(U(r, s)\times U(u, v))$ and to extend it so as to cover all the above systems. Through the H-gauge transformation we obtain higher-order non-linear systems which enable us to gauge connect a large number of specific equations known in the literature. On concrete models we also demonstrate how to exploit the gauge relations to find the inverse scattering data, Jost functions and soliton solutions of σ model systems.

The paper is organised as follows. In § 2 the general scheme of gauge equivalence between non-linear systems is outlined and applied to establish the relation between LLE and NLSE with a non-compact manifold. Particular examples yield known as well as new results. The H-gauge transformation generates some higher-order special types of equations. In § 3 we find the generalised LLE-type systems equivalent to derivative and mixed NLSE and represent them as different orbits with non-degenerate Poisson brackets. Through the H-gauge transformation we generate here a number of higher-order equations. Section 4 demonstrates, in particular cases of attractive and repulsive scalar NLSE, the applicability of gauge equivalence for finding soliton solutions and other information. Section 5 deals with the non-linear σ model with non-compact manifold and its equivalent generalised sine-sinh-Gordon equations. Section 6 contains the conclusions.

2. LLE with non-compact Grassmannian manifold and gauge equivalent NLSE

It is well known that a given integrable non-linear system is related to a linear one $\Phi_x = U\Phi$, $\Phi_t = V\Phi$, the compatibility condition of which yields the given non-linear equation. Under the local gauge transformation $g(x, t) \in \mathrm{G}$, G being a Lie group, the Jost function changes:

$$\Phi \to \Phi' = g^{-1}\Phi, \qquad \Phi'_x = U'\Phi', \qquad \Phi'_t = V'\Phi'. \tag{2.1}$$

For $\lambda = \lambda_0$ belonging to the continuous spectrum, $\Phi \in \mathrm{G}$ and $U, V \in \mathscr{G}$, $\mathscr{G}$ being the corresponding Lie algebra. We may fix the gauge element by choosing $g(x, t; \lambda_0) = \Phi(x, t; \lambda)|_{\lambda=\lambda_0} \in \mathrm{G}$ which gives

$$U' = g^{-1}Ug - g^{-1}g_x = g^{-1}(U - U_0)g, \qquad U(x, t; \lambda_0) = U_0 \tag{2.2a}$$

$$V' = g^{-1}Vg - g^{-1}g_t = g^{-1}(V - V_0)g, \qquad V(x, t; \lambda_0) = V_0 \tag{2.2b}$$

with the new gauge equivalent equation as

$$U'_t - V'_x + [U', V'] = g^{-1}\{(U_t - V_x + [U, V]) - (U_{0t} - V_{0x} + [U_0, V_0])\}g = 0. \tag{2.3}$$

If the original non-linear equation is an integrable system which can be solved by using inverse scattering theory with scattering matrix T given by $\Phi_- = \Phi_+ T$, where $\Phi_\pm$ are Jost functions with asymptotes given at $\pm\infty$, then under (2.1) the scattering matrix of the gauge transformed system remains unchanged

$$T' = \Phi'^{-1}_+\tilde{\Phi}'_- = \Phi'^{-1}_+ T_0 \Phi'_- = \Phi'^{-1}_+ T_0 g_-^{-1}\Phi_- = \Phi'^{-1} g_+^{-1}\Phi_- = T \tag{2.4}$$

where $g_\pm = \Phi_\pm(\lambda = \lambda_0)$ and $T_0 = T(\lambda = \lambda_0)$. This helps us to avoid the IST investigation of a system if we know the corresponding information on its gauge-connected counterpart. The integrability property is also obviously preserved under such transformations. Note that since only for $\lambda = \lambda_0$ chosen from continuous spectrum Φ has fixed group properties, the gauge elements $g \in \mathrm{G}$ must be taken only at $\lambda = \lambda_0$. Since the domain of λ_0 may differ for compact and non-compact cases, depending on the boundary conditions, its value should be chosen carefully. This fact will be illustrated later with concrete examples.

We propose a generalised Landau-Lifshitz equation with non-compact manifold:

$$S_t = \frac{1}{2\mathrm{i}}[S, S_{xx}]; \qquad S \in M = \mathrm{G/H}, \qquad \mathrm{G} = \mathrm{SU}(p, q),$$
$$\mathrm{H} = \mathrm{S}(U(u, v) \times U(r, s)) \tag{2.5}$$

where $u+v=m$, $s+t=n$, $u+r=p$, $v+s=q$, $p+q=m+n=N$, and find its gauge equivalent system using the above scheme. The linear system corresponding to (2.5) is given by

$$U = \mathrm{i}\lambda S, \qquad V = 2\mathrm{i}\delta^2\lambda^2 S + \tfrac{1}{2}\lambda[S, S_x], \qquad \delta = (m+n)/2mn \tag{2.6}$$

with S satisfying

$$S^2 = aI + bS, \qquad \bar{S} = \Gamma S^{+}\Gamma = S \tag{2.7}$$

where

$$\Gamma = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}, \qquad \Gamma_1 = \begin{pmatrix} I_u & 0 \\ 0 & -I_v \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix} \tag{2.8}$$

a, b are constants and I_i a unit $i \times i$ matrix. G/H being a symmetric space it is always possible to express $S = g\Sigma g^{-1}$, $g \in \mathrm{G}$ and

$$\Sigma = \begin{pmatrix} I_m/m & 0 \\ 0 & -I_n/n \end{pmatrix}.$$

Hence we get

$$\begin{aligned} &S_\mu = g[\Sigma, L_\mu]g^{-1}, \qquad L_\mu = g^{-1}g_\mu \\ &\tfrac{1}{2}[S, S_x] = 2g\Sigma[\Sigma, L_1]g^{-1} \end{aligned} \tag{2.9}$$

which gives the gauge transformed operators in the form

$$U' = g^{-1}Ug = g^{-1}g_x = \mathrm{i}\lambda\Sigma + A, \qquad L_1 \equiv A \tag{2.10a}$$
$$V' = g^{-1}Vg - g^{-1}g_t = 2\mathrm{i}\delta^2\lambda^2\Sigma + \lambda\Sigma[\Sigma, A] + B \qquad L_0 \equiv B. \tag{2.10b}$$

Taking A and B in the explicit form

$$A = \begin{pmatrix} A_1 & \bar{\psi} \\ -\psi & A_2 \end{pmatrix}, \qquad B = \mathrm{i}\delta\begin{pmatrix} b_1 & \bar{\psi}_x + \bar{\psi}A_2 - A_1\bar{\psi} \\ \psi_x + \psi A_1 - A_2\psi & b_2 \end{pmatrix} \tag{2.11}$$

with $\bar{\psi} = \Gamma_1\psi^{+}\Gamma_2$, $\bar{A}_i = -A_i$, $\mathrm{Tr}\,(A_1 + A_2) = 0$, $A_i \equiv H^i_x$ and $b_1 = (\psi\bar{\psi} - \mu I_m + H^1_t$, $b_2 = -(\psi\bar{\psi} - \mu(m/n)I_n) + H^2_t$. The compatibility condition of (2.10) yields directly the following matrix NLSE

$$\mathrm{i}\psi_t + \psi_{xx} + 2(\psi\bar{\psi}\psi - \mu\psi) + 4\mathrm{i}\delta\lambda_0\psi_x = 0. \tag{2.12}$$

We have set $A_1 = \mathrm{i}\lambda_0 I_m/m$ and $A_2 = -\mathrm{i}\lambda_0 I_n/n$ in deducing (2.12), while more general cases will be discussed below. Note that for trivial boundary conditions λ_0 is arbitrary real and may be trivial (Zakharov and Takhtajan 1979), but for a non-trivial boundary condition which is important in non-compact cases λ_0 is non-trivial. Under the transformation $\psi' = R_2\psi\bar{R}_1$ (2.12) is covariant if

$$\psi'\bar{\psi}'\psi' = R_2\psi(\bar{R}_1R_1)\bar{\psi}(\bar{R}_2R_2)\psi\bar{R}_1 = R_2(\psi\bar{\psi}\psi)\bar{R}_1 \tag{2.13}$$

i.e. when

$$\bar{R}_1R_1 = \bar{R}_2R_2 = I \qquad \text{or} \qquad R_1 \in U(u, v), \qquad R_2 \in U(r, s). \tag{2.14}$$

We therefore conclude that LLE (2.5) with non-compact Grassmannian manifold $SU(p,q)/S(U(u,v)\times U(r,s))$ is gauge equivalent to matrix NLSE (2.12) with $\mathscr{P}=U(u,v)\times U(r,s)$ as the internal symmetry group. The number of independent field variables ψ, $\bar\psi$ is $2mn$ which also coincides with

$$\dim M=\dim \mathrm{G}-\dim \mathrm{H}=[(m+n)^2-1]-(m^2+n^2-1)=2mn$$

showing the number of independent fields in LLE.

2.1. *Examples*

Let us consider (2.5) with $u=m$, $v=0$ and $r=p-m>0$, i.e. with

$$S\in SU(p,q)/S(U(m)\times U(p-m,q)) \tag{2.15}$$

and the following particular cases.

(i) $p=N$, $q=0$, $\Gamma_1=\Gamma_2\to I$ hence $\bar S=S^+$ which recovers the equivalence of LLE with compact manifold $S\in SU(N)/S(U(m)\times U(n))$ and the matrix NLSE (MNLS) (Honnerkamp 1981). $m=1$ yields LLE with $S\in CP^N$ and gauge connected vector NLSE (VNLSE) of attractive type (Orfanidis 1980). $N=1$ gives the equivalence of standard LLE with $S\in S^2$ and the scalar NLSE (SNLS) of attractive type (Zakharov and Takhtajan 1979). Thus we recover the results known previously and moreover get the following results from the relevance of non-compactness.

(ii) $p=m$, $q=n$ gives $\Gamma_1=I$, $\Gamma_2=-I$ and $\bar\psi=-\psi^+$. That is LLE with $SU(p,q)/S(U(p)\times U(q))$ is gauge equivalent to MNLS of repulsive type

$$\mathrm{i}\hat\psi_t+\hat\psi_{xx}-2(\hat\psi\hat\psi^+\hat\psi-\mu\hat\psi)=0. \tag{2.16}$$

$p=1$ connects LLE with $SU(1,N-1)/U(N-1)$ and VNLS of repulsive type

$$\mathrm{i}\boldsymbol{\psi}_t+\boldsymbol{\psi}_{xx}-2\left(\sum_{a=1}^{N}|\psi^a|^2-\mu\right)\boldsymbol{\psi}=0. \tag{2.17}$$

$N=2$ gives a $SU(1,1)/U(1)$ version of LLE and related repulsive SNLS (Kundu 1982).

(iii) $m=1$ reduces (2.15) to LLE with non-compact manifold $SU(p,q)/U(p-1,q)$ and a $\mathscr{P}=U(p-1,q)$ 'attractive-repulsive' VNLSE

$$\mathrm{i}\boldsymbol{\psi}_t+\boldsymbol{\psi}_{xx}+2\left(\sum_{a=1}^{p-1}|\psi^a|^2-\sum_{b=1}^{q}|\psi^b|^2-\mu\right)\boldsymbol{\psi}=0 \tag{2.18}$$

which in limiting cases leads again to VNSLE of attractive or repulsive type. The above equivalence is schematically depicted in figure 1.

2.2. *Extension of gauge equivalence to higher-order and special types of* NLSE

It is not difficult to check that from a standard NLSE (2.12) through local gauge transformation $\psi\to h_2^{-1}\psi h_1$ relative to group element

$$h=\begin{pmatrix} h_1 & 0\\ 0 & h_2\end{pmatrix}\in \mathrm{H}=S(U(u,v)\times V(s,t))$$

one gets a new higher-order equation:

$$\begin{aligned}\mathrm{i}\psi_t+\psi_{xx}&+2(\psi\bar\psi\psi-\mu\psi)+4\mathrm{i}\lambda_0\delta\psi_x-[(\mathrm{i}/\delta)(\psi H_t^1-H_t^2\psi)\\ &+2(\psi_xH_x^1-H_x^2\psi_x)+(\psi H_{xx}^1-H_{xx}^2\psi)\\ &+2(H_x^2\psi H_x^1)]+(H_x^2)^2\psi+\psi(H_x^1)^2=0\end{aligned} \tag{2.19}$$

where $H_\mu^i=h_i^{-1}\,\partial_\mu h_i$ corresponding to the linear system (2.11) in the general case.

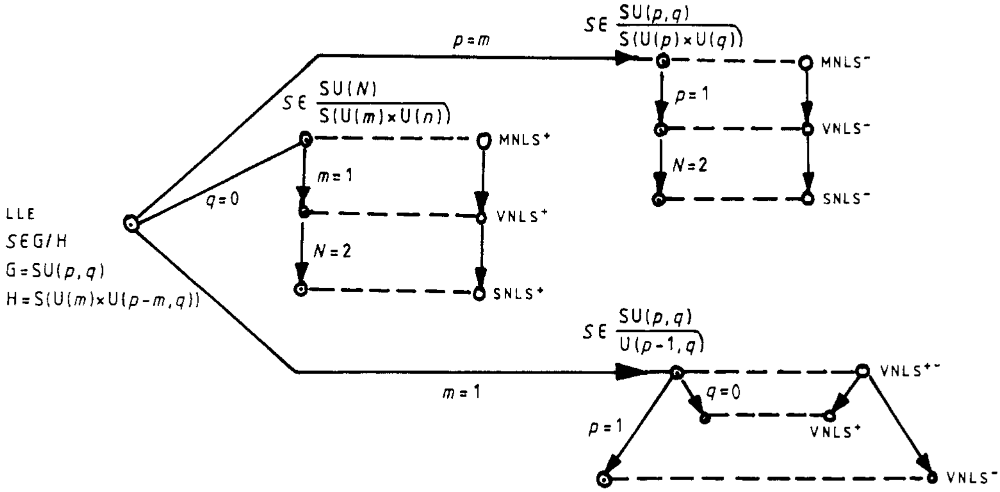

Figure 1. Gauge equivalence between generalised LLE with non-compact Grassmannian manifold and NLSE with their different reductions. (±) represent 'attractive' or 'repulsive' while (+ −) represents 'attractive-repulsive' type cases.

Since under $h(x, t) \in H$ gauge transformation, the σ model field is invariant

$$S' = g'\Sigma g'^{-1} = gh\Sigma h^{-1}g^{-1} = g\Sigma g^{-1} = S \tag{2.20}$$

the LLE system remains unchanged under such transformations and consequently along with NLSE (2.12) it is equivalent to all of its transformed forms like (2.19). We consider two interesting particular cases of (2.19) setting $m = n = 1$ for convenience, i.e. $h \in U(1)$ and $-H^1_\mu = H^2_\mu = i\theta_\mu$.

(i) The case $\theta = \theta(x)$, $a = \theta_x$ being an arbitrary smooth function of x gives NLSE with x dependent coefficients

$$i\psi_t + \psi_{xx} \pm 2(|\psi|^2 - \mu)\psi + 4ia\psi_x + 2(ia_x - 2a^2)\psi = 0. \tag{2.21}$$

(ii) The choice of θ_μ as 'particle' and 'current' densities

$$\theta_x = \mp\delta|\psi|^2, \qquad \theta_t = \pm i\delta(\psi\psi^*_x - \psi^*\psi_x) \tag{2.22}$$

on the other hand reduces (2.19) to a higher-order system (HNLS)

$$i\psi_t + \psi_{xx} \pm 2|\psi|^2\psi + 4\delta^2|\psi|^4\psi \pm 4i\delta(|\psi|^2)_x\psi = 0. \tag{2.23}$$

Note that (2.22) is compatable ($\theta_{xt} = \theta_{tx}$) due to equation (2.23).

It may also be shown that (Kundu and Pashaev 1983) SU(2) LLE with easy axis ($\Delta > 0$) anisotropy (ALLE) is gauge equivalent to attractive SNLS and the easy plane case ($\Delta < 0$) to Zakharov-AKNS system, while the anisotropic SU(1, 1) LLE is equivalent to repulsive SNLS.

3. Extended LLE and their equivalent NLS type equations

Using a similar technique to that applied above for standard NLSE, we may also find the extended LLE, gauge equivalent to various NLS type equations (Kundu 1984). In particular scalar derivative NLSE(DNS)

$$q_t + q_{xx} \pm i\alpha(|q|^2 q)_x = 0 \tag{3.1}$$

is equivalent to extended LLE(DLL)

$$S_t + (1/2i)[S, S_{xx}] - 4\alpha\lambda_0^2 S_x + (1/4\alpha\lambda_0^2)S_x^3 = 0, \qquad \alpha > 0. \tag{3.2}$$

On the other hand mixed scalar NLSE(XNS)

$$iq_t + q_{xx} \pm \beta|q|^2 q \pm i\alpha(|q|^2 q)_x = 0, \qquad \alpha > 0, \beta > 0 \tag{3.3}$$

is gauge related to a new integrable system (XLL)

$$S_t + (\varepsilon/2i)[S, S_{xx}] + \gamma S_x + \rho S_x^3 + id[S_t, S_x] + d[SS_x, S_{xx}] = 0 \tag{3.4}$$

where ε, γ, ρ and d are different constants depending on λ_0, α and β. The $+(-)$ sign in (3.1) and (3.3) corresponds to $S \in SU(2)/U(1)(SU(1,1)/U(1))$ in (3.2) and (3.4). Parameters in (3.4) are simplified in the following limiting cases

$$d = 0, \qquad \varepsilon = 1, \qquad \rho = 0, \qquad \gamma = 4(2\beta)^{1/2}\lambda_0, \qquad \text{for } \alpha = 0, \beta \neq 0 \tag{3.5a}$$

$$d = 0, \qquad \varepsilon = 1, \qquad \rho = 1/4\alpha\lambda_0^2, \qquad \gamma = -4\alpha\lambda_0^2, \qquad \text{for } \alpha \neq 0, \beta = 0. \tag{3.5b}$$

Note that (3.6*a*) leads to standard LLE while (3.6*b*) leads to DLL (3.2).

3.1. *Higher-order NLS type systems*

Similarly to the U(1) gauge transformation discussed in § 2.2 for NLSE, the choice

$$\theta_x = \mp\delta|q|^2, \qquad \theta_t = \pm i\delta(qq_x^* - q^* q_x) + \tfrac{3}{2}\alpha\delta|q|^4 \tag{3.6}$$

leads XNS (3.3) to a new HXNS

$$iQ_t + Q_{xx} \pm i\alpha(|Q|^2 Q)_x \pm \beta|Q|^2 Q + \delta(4\delta + \alpha)|Q|^4 Q \pm 4i\delta(|Q|^2)_x Q = 0. \tag{3.7}$$

The limiting case $\alpha = 0$, $\beta \neq 0$ recovers HNLS (2.23) while the other extreme $\alpha \neq 0$, $\beta = 0$ generates higher-order derivative NLS(HDNS)

$$iQ_t + Q_{xx} \pm i\alpha(|Q|^2 Q)_x + \delta(4\delta + \alpha)|Q|^4 Q \pm 4i\delta(|Q|^2)_x Q = 0. \tag{3.8}$$

It is remarkable that the particular choice $\alpha = -4\delta$ leads it further to the Chen-Lee-Liu (CLL) equation

$$iQ_t + Q_{xx} \pm i\alpha|Q|^2 Q_x = 0 \tag{3.9}$$

whereas the choice $\alpha = -2\delta$ gives a Gerdjikov-Ivanov type one (GI1) equation

$$iQ_t + Q_{xx} + \tfrac{1}{2}\alpha^2|Q|^4 Q \mp i\alpha Q^2 Q_x^* = 0. \tag{3.10}$$

If however we set $\alpha = -4\delta$, $\beta \neq 0$, (3.7) yields CLL-NS

$$iQ_t + Q_{xx} \pm \beta|Q|^2 Q \pm i\alpha|Q|^2 Q_x = 0 \tag{3.11}$$

while for $\alpha = -2\delta$, $\beta \neq 0$ we get the Gerdjikov-Ivanov type two (GI2) equation

$$iQ_t + Q_{xx} \pm \beta|Q|^2 Q + \tfrac{1}{2}\alpha|Q|^4 Q \mp i\alpha Q^2 Q_x^* = 0. \tag{3.12}$$

Therefore all equations (3.8)-(3.10) are naturally gauge connected with DNS while (3.11), (3.12) and (3.7) are related to XNS.

3.2. *Representation of* LLE *and its extensions as orbits*

Non-degenerate Poisson brackets for functions on G* are associated with orbits of the algebra action in it (Faddeev 1983). Hence for a function

$$U_a(\lambda) = \sum \xi_a^m \lambda^{-1-m} \tag{3.13}$$

the Poisson brackets give

$$\{\xi_a^m, \xi_b^n\} = c_{ab}^c \xi_c^{m+n}, \qquad n, m \geqslant 0 \tag{3.14}$$

c_{ab}^c being the structure constant. We demonstrate that in the light of the above general approach the LLE and its extensions may be defined as different orbits.

For SU(2)(SU(1, 1)) LEE we may set $\xi_a^0 = S^a(x)$, $\xi_a^m = 0$, $m \geqslant 1$ which gives

$$\{\xi_a^0(x), \xi_b^0(y)\} = \mathrm{i} f_{\ c}^{ab} \xi_c^0(x)\, \delta(x-y) \tag{3.15}$$

and $\xi^2 = \Sigma\, g_{ab} S^a(x) S^b(x) = 1$ defining an orbit. In the case of DLLE (3.3) identifying $\xi_a^0 = S^a(x)$, $\xi_a^1 = \mathrm{i} f^{abc} S^b S_x^c$ and $\xi_a^m = 0$, $m \geqslant 2$ we get the algebra

$$\xi^{02} = 1, \qquad g_{ab}\xi_a^0 \xi_b^1 = S^a f^{abc} S^b S_x^c = 0 \tag{3.16a}$$

$$\{\xi_a^0(x), \xi_b^0(y)\} = \mathrm{i} f^{ab}{}_c \xi_c^0\, \delta(x-y) \tag{3.16b}$$

$$\begin{aligned} \{\xi_a^0(x), \xi_b^1(y)\} &= \mathrm{i}\{S^a, S^c S_y^d\} f^{bcd} \\ &= \mathrm{i}(\{S^a, S^c\} S_y^d + \{S^a, S_y^d\} S^c) f^{bcd} \\ &= f^{ab}{}_c \xi_c^1(y)\, \delta(x-y) \end{aligned} \tag{3.16c}$$

and similarly

$$\{\xi_a^1(x), \xi_b^1(x)\} = 0 \tag{3.16d}$$

exactly agreeing with the formula (3.14). The case of XLL can also be similarly treated. In figure 2 we schematically depict the relation between different NLS and LLE systems found above.

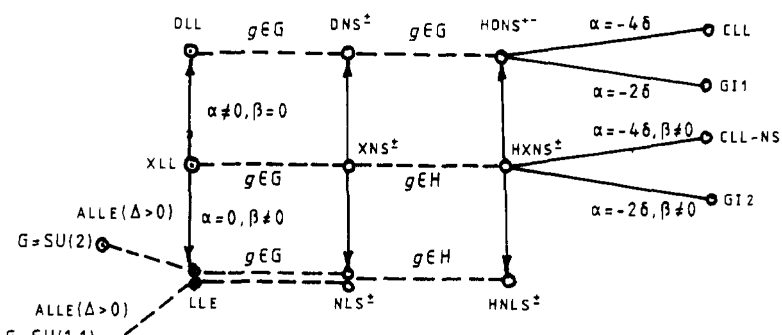

Figure 2. Extended gauge equivalence scheme of different LLE and NLS type equations. Here H = U(1) and G = SU(2) or SU(1, 1) correspond to attractive (+) or repulsive (−) cases, respectively.

4. Construction of LLE solutions from NLSE

In earlier research work the main emphasis has been given to establishing the gauge equivalence between different dynamical systems without paying much attention to the utilisation of such a beautiful relationship. We aim to demonstrate here some applications of such an equivalence. In particular, we use the inverse scattering data of NLSE which has been well investigated to explore such information for the LLE model, the individual study of which is sometimes very tedious. As shown in § 2, the Jost function of LLE Φ'_α may be expressed through that of NLSE Φ_α as

$$\Phi'_\alpha = \Phi_{0\alpha}^{-1}\Phi_\alpha$$

where

$$\Phi_{0\alpha} = \Phi_\alpha(\lambda = \lambda_0) \qquad \text{and} \qquad \Phi_\alpha = (\phi_\alpha, \tilde{\phi}_\alpha)$$

with

$$\phi_\alpha = \begin{pmatrix} \phi_\alpha^1 \\ \phi_\alpha^2 \end{pmatrix}, \qquad \tilde{\phi}_\alpha = \begin{pmatrix} \varepsilon\phi_\alpha^{2*} \\ \phi_\alpha^{1*} \end{pmatrix}.$$

ϕ_α, $\tilde{\phi}_\alpha$ form the basis in the space of Jost solutions of the spectral problem connected with NLSE. The subscript $\alpha = \pm$ indicates that the asymptotes defined at $x \to \pm\infty$ and $\varepsilon = \mp 1$ correspond to NLSE of attractive and repulsive type, respectively, relating to LLE with $S \in \mathrm{SU}(2)/\mathrm{U}(1)$ and $\mathrm{SU}(1,1)/\mathrm{U}(1)$ manifolds. As already shown in § 2 the scattering matrices of the gauge equivalent LLE and NLSE are identical. The field solution of LLE may be expressed through NLSE Jost functions as

$$S_+ = \Phi_{0+}^{-1}\sigma_3\Phi_{0+} = T_0(\Phi_{0-}^{-1}\sigma_3\Phi_{0-})T_0^{-1} = T_0 S_- T_0^{-1}. \tag{4.1}$$

Both $S_\pm$ are LLE solutions. We take for definiteness $S = S_+$ and obtain

$$S = \frac{1}{\Delta_0}\begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \qquad \begin{aligned} S_3 &= |\phi_0^1|^2 + \varepsilon|\Phi_0^2|^2 \\ S^+ &= -2\phi_0^1\phi_0^2 \\ S^- &= -\varepsilon S^{+*}, \qquad \Delta = \det\Phi_0. \end{aligned} \tag{4.2}$$

4.1. *NLSE of attractive type and SU(2)/U(1) LLE*

NLSE of the attractive type have been well investigated using the inverse scattering method (Zakharov and Sabat 1971). The corresponding Jost function for the N-soliton solution is given by

$$\tilde{\phi}(x,\lambda) = \exp(-\mathrm{i}\lambda x)\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{n=1}^{N} \frac{c_n\phi_n \exp(\mathrm{i}\lambda_n x)}{(\lambda - \lambda_n)}\right) \tag{4.3}$$

where $\phi_n = \phi(x, \lambda_n)$ and λ_n is the discrete spectrum. For simplicity we consider $N = 1$. Hence the one-soliton solution of NLSE is

$$\psi(x) = -2\sum_n^N c_n\phi_n^1(x)\exp(\mathrm{i}\lambda_n x)|_{N=1} = -2\mathrm{i}\eta\exp(\mathrm{i}\gamma)\,\mathrm{sech}\,y \tag{4.4}$$

where $\lambda_1 = \xi + \mathrm{i}\eta$, $\gamma = 2\xi x + 4(\xi^2 - \eta^2)t + \phi_0$, and $y = 2\eta(x - x_0 - 4\xi t)$.

On the other hand from (4.3) one obtains

$$\begin{aligned}\phi_0^1 &= -\rho(\lambda_0)\exp[\mathrm{i}(\lambda-\lambda_0 x)]\operatorname{sech} y\\ \phi_0^2 &= [1-\rho(\lambda_0)\exp(-y)\operatorname{sech} y]\exp(-\mathrm{i}\lambda_0 x)\end{aligned} \tag{4.5}$$

with $\rho(\lambda_0)=\mathrm{i}\eta/(\lambda-\lambda_0)$, which gives directly from (4.2) the LLE soliton solution

$$S_3 = 1-2|\rho|^2\operatorname{sech}^2 y$$

and

$$\arg(S^+) = 2\lambda_0 x - \gamma - \tan^{-1}\{[(\lambda_0-\xi)/\eta]\coth y\}. \tag{4.6}$$

Note that for $\lambda_0=0$ (4.6) coincides with the solution found by Takhtajan (1977) through the direct IST scheme.

4.2. NLSE *of repulsive type and non-compact SU(1, 1)/U(1)* LLE

The inverse scattering programme for repulsive type NLSE becomes complicated due to a non-trivial boundary condition† $\lim_{|x|\to\infty}|\psi|^2\to\mu^2$. Here $\xi=\pm(\lambda^2-\mu^2)^{1/2}$ actually serves the role of spectral parameter and $\xi(\lambda)$ is defined on a two-sheeted Riemann surface with cuts at $(-\infty,-\mu)$ and (μ,∞). The bound states are given at $\xi_n=(\lambda_n^2-\mu^2)^{1/2}=\mathrm{i}\nu_n$ with $|\lambda_n|^2<\mu^2$. The Jost solution corresponding to the one-soliton solution is given by (Zakharov and Shabat 1973)

$$\phi=\exp(-\mathrm{i}\xi x)\begin{pmatrix}1 & \\ & \lambda-\xi\end{pmatrix}\left[\begin{pmatrix}1\\1\end{pmatrix}-\frac{1}{\nu+\mathrm{i}\xi}\begin{pmatrix}K_1 & K_2\\ K_2^* & K_1^*\end{pmatrix}\right], \tag{4.7}$$

where $K_1(y)=\nu f(y)$, $K_2(y)=\nu(\gamma-\mathrm{i}\nu)f(y)$, $f(y)=[1+\exp(2y)]^{-1}$, $\xi^2=\lambda^2-1$, $\lambda_1=\gamma$, $\nu^2=1-\gamma^2$ and $y=\nu(x-x_0-2\gamma t)$. The soliton solution to repulsive type NLSE is

$$\psi(x,t)=[(\gamma+\mathrm{i}\nu)^2+\exp(2y)]f(y). \tag{4.8}$$

The Hamiltonian of the gauge equivalent non-compact model may be given by

$$H=\int_{-\infty}^{\infty}\operatorname{Tr}(S_x^2)\,\mathrm{d}x=\frac{1}{2}\int_{-\infty}^{\infty}(S_x^{3^2}-S_x^{2^2}-S_x^{1^2})\,\mathrm{d}x \tag{4.9}$$

with $S^{3^2}-S^{2^2}-S^{1^2}=1$, $S=S^a\tau_a\in\mathrm{SU}(1,1)/\mathrm{U}(1)$. The soliton solution to the corresponding field equation may be easily obtained from (4.2) with $\varepsilon=1$ and using (4.7). Note that since λ_0 must be from a continuous spectrum, we have the restriction $|\lambda_0|>\mu=1$. For the choice $\lambda_0-\xi_0=\gamma$ we get the solution

$$\begin{aligned}S_3 &= 1+(2\gamma^2/\nu^2)\tanh^2 y\\ \arg(S^+) &= 2\xi x+\tan^{-1}[(\gamma/\nu)\tanh y].\end{aligned} \tag{4.10}$$

In this case the soliton velocity $u\equiv 2\gamma=2(\lambda_0-\xi_0)$ is restricted, $2>u\geqslant 0$, though for general choice of λ_0 the velocity is arbitrary.

5. Non-linear σ model with non-compact Grassmannian manifold and the gauge connected equations

We apply here the ideology of gauge equivalence to the non-linear σ model theory (Honnerkamp and Eichenherr 1981). The linear system of the equivalent generalised

† Trivial conditions exhibit no discrete spectrum and hence no regular soliton solution.

'sine-sinh-Gordon' type equation may be given by

$$U=\lambda C+A, \qquad V=\lambda^{-1}B, \qquad U, V\in \mathrm{su}(p,q). \tag{5.1}$$

For $m=1$ the structure of the operators is of the form

$$A=\left(\begin{array}{c|cc} \mathrm{i}c_0 & & \\ \hline & -\mathrm{i}c_0 & -\bar{c} \\ & \bar{c} & 0 \end{array}\right) \qquad B=\left(\begin{array}{c|c} 0 & -\bar{\varphi} \\ \hline \varphi & 0 \end{array}\right), \qquad C=\left(\begin{array}{c|cc} 0 & -1 & \mathbf{0} \\ \hline 1 & & 0 \\ \mathbf{0} & & \end{array}\right) \tag{5.2}$$

with

$$\begin{aligned} \bar{c}&=(c_1^*,\ldots,c_{p-2}^*,\mathrm{i}c_{p-1}^*,\ldots,\mathrm{i}c_{N-2}^*) \\ \bar{\varphi}&=(\varphi_0^*,\varphi_1^*,\ldots,\varphi_{p-2}^*,\mathrm{i}\varphi_{p-1}^*,\ldots,\mathrm{i}\varphi_{N-2}^*) \end{aligned} \tag{5.3}$$

and $\Sigma_{a=0}^{p-2}|\varphi_a|^2-\Sigma_{b=p-1}^{N-2}|\varphi_b|^2=1$. The compatability condition of (5.1) gives the relation

$$\begin{aligned} \mathrm{i}c_0&=-\left(\partial_\xi\varphi_0\varphi_0^*+\sum_{\alpha=1}^{p-2}\partial_\xi\varphi_\alpha^*\varphi_\alpha-\sum_{b=p-1}^{N-2}\partial_\xi\varphi_b^*\varphi_b\right)(3|\varphi_0|^2-1)^{-1} \\ c_k&=(\partial_\xi\varphi_k+\mathrm{i}c_0\varphi_k)\varphi_0^{-1}, \qquad k=1,\ldots,N-2 \end{aligned} \tag{5.4}$$

and the equation of motion

$$\mathrm{i}\partial_\eta c_0=\varphi_0-\varphi_0^*, \qquad \partial_\eta c_k=-\varphi_k. \tag{5.5}$$

The gauge equivalent non-linear σ model is given by

$$S_{\xi\eta}+\tfrac{1}{2}(S_\xi S_\eta+S_\eta S_\xi)S=0, \qquad S^2=1 \tag{5.6}$$

with $S\in\mathrm{SU}(p,q)/\mathrm{U}(p-1,q)$. The limiting case $p=N$ recovers the known compact $cp^N\sigma$ model. The case $p=2$, $q=1$ reduces (5.3) to $|\varphi_1|^2-|\varphi_2|^2=1$. Hence choosing $\varphi_1=\cosh\beta\exp(\mathrm{i}\alpha)$ and $\varphi_2=\sinh\beta\exp(\mathrm{i}\psi)$ we get the sine-sinh-Gordon equation

$$\begin{aligned} \beta_{\xi\eta}&=\sinh\beta[(\psi_\xi-\alpha_\eta)(\psi_\xi+c_0)/\cosh\beta-\cos\alpha] \\ c_{0\eta}&=2\cosh\beta\sin\alpha \\ \psi_{\xi\eta}\tanh\beta&=-[3\sinh\beta\sin\alpha+(c_0+\psi_\xi)\cosh^{-2}\beta+\beta_\xi(\psi_\eta-\alpha_n)] \end{aligned} \tag{5.7}$$

with $c_0=(\tanh^2\beta\psi_\xi-\alpha_\eta)(2+\tanh^2\beta)^{-1}$ equivalent to the non-linear σ model with $S\in\mathrm{SU}(2,1)/\mathrm{U}(1,1)$. In the limiting case $\alpha=\psi=0$, $c_0=0$ answering to the O(2, 1) subgroup of U(2, 1), (5.7) reduces to the sinh-Gordon equation, while for $\beta=0$, $c_0=-\alpha_\xi/2$ to the sine-Gordon equation. $p=1$, $q=1$ gives also the sinh-Gordon equation

$$\beta_{\xi\eta}+4\sinh\beta=0$$

and the gauge related σ model with SU(1, 1)/U(1). The $p=2$, $q=0$ case recovers the gauge equivalence of the sine-Gordon and SU(2)/U(1) σ models.

6. Conclusions

We have shown gauge equivalence between generalised LLE with non-compact manifolds and matrix NLSE. Such generalisations, apart from their mathematical beauty, may also have physical applications in describing spins of different sorts or multichained magnetic crystals with interchain coupling (Ovchinnikov and Onoshuk 1978, Kundu and Kundu 1983). The coupling constants of different signs may be described by non-compact model structures. Extended LLE gauge equivalence to DNS and XNS also presents new integrable spin models. A higher-order NLS type equation gauge connects a number of known equations and also generates new integrable systems. Applications of gauge equivalence for finding soliton solutions and other IST information show that using the information about any system one can, in principle, solve a large number of gauge connected equations which manifests the importance of establishing such gauge relations.

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